Computing the Geometric Intersection Number of Curves

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Problem

(a) Number of crossings: too many!
(b) Number of crossings: 1 → optimal

Three problems:
- Deciding if a curve can be made simple by homotopy.
- Finding the minimum possible number of self-intersections.
- Finding a corresponding minimal representative.
An Old Problem

- Poincaré, 5ème complément analysis situs (1905)
- Reinhart, Algorithms for jordan curves on compact surfaces (1962)
- Chillingworth, Simple closed curves on surfaces (1969)
- Cohen and Lustig, Paths of geodesics and geometric intersection numbers (1987)
- Gonçalves et al., An algorithm for minimal number of (self-)intersection points of curves on surfaces (2005)
**INPUT:**
A combinatorial surface of complexity \( n \).
A closed walk of length \( l \).

A choice of edge orders leads to a generic perturbation of the curve.

**Discrete vs. Continuous**

- Each equivalence class of curves can be described by a closed walk.
- Each minimal configuration can be realized by a closed walk with appropriate orders on the edges.
de Graaf and Schrijver’s Algorithm

Reidemeister moves:

de Graaf and Schrijver (1997)

Every curve can be made minimally crossing by Reidemeister moves.
System of Quads
Canonical Representative

Canonical representative:

→ Choose a shortest representative and push it to the right as much as possible.

Erickson and Whittlesey (2013)

The canonical representative of a curve is its only representative with no spur, no bracket, no angle -1 and not all angles -2.
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Lazarus and Rivaud (2012)

After $O(n)$ precomputation time, one can compute the canonical representative of a curve of length $\ell$ in $O(\ell)$ time. Its length is at most $2\ell$. 
Properties of the Canonical Form

Gersten and Short (1990)

A nontrivial contractible closed curve on a system of quads must have either a spur or four brackets.

Lemma

A curve in canonical form has no monogon.

Lemma

A curve in canonical form has only flat bigons (i.e. the two sides of each bigon correspond to the same path in the system of quads).
Monogons and Bigons

Hass and Scott (1985)

If a curve has excess self-intersections then it has a **singular** monogon or a **singular** bigon.
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If a curve has excess self-intersections then it has a **singular** monogon or a **singular** bigon.
Computing a Minimal Representative

Algorithm for **primitive** curves:

- Compute a canonical representative.
- Choose a random order on the edges.
- Look for a singular bigon ($O(\ell^2)$):
  - If there is one, exchange its two paths to remove intersections.
  - Repeat until there is no more singular bigon.

**Theorem**

This algorithm computes a minimal representative in $O(\ell^4)$ time.
Non-Primitive Curves

Formula

Let $c$ be a curve, $p \in \mathbb{N}^*$. Then,

$$i(c^p) = p^2 \cdot i(c) + p - 1$$

where $i(x)$ is the geometric intersection number of curve $x$. 
Double Paths

Lemma

The maximal double paths form a partition of the pairs of “identical” edges of a given curve. So we can compute the number of crossing double paths in $O(\ell^2)$ time.

Cohen and Lustig (1987)

Let $\Sigma$ be a combinatorial surface whose faces are all perforated. Let $c$ be a closed walk without spur in $\Sigma$, then $i(c)$ is its number of crossing double paths.
Main Result

Theorem

Given a curve \( c \) of length \( \ell \) on a surface of complexity \( n \), one can compute the geometric intersection number of \( c \) in \( O(n + \ell^2) \) time.
Main Result

Theorem

Given a curve $c$ of length $\ell$ on a surface of complexity $n$, one can compute the geometric intersection number of $c$ in $O(n + \ell^2)$ time.

⇒ We first compute the canonical representative of $c$ in $O(n + \ell)$ time.
Main Result

**Theorem**

Given a curve $c$ of length $\ell$ on a surface of complexity $n$, one can compute the geometric intersection number of $c$ in $O(n + \ell^2)$ time.

- We first compute the canonical representative of $c$ in $O(n + \ell)$ time.
- We compute $i(c)$ in the surface with all its faces perforated in $O(\ell^2)$ time.
Main Result

**Theorem**

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- We first compute the canonical representative of \( c \) in \( O(n + \ell) \) time.
- We compute \( i(c) \) in the surface with all its faces perforated in \( O(\ell^2) \) time.
- The minimum \( i(c) \) is attained by some orders on the edges. It has no bigon.
Main Result

Theorem

Given a curve $c$ of length $\ell$ on a surface of complexity $n$, one can compute the geometric intersection number of $c$ in $O(n + \ell^2)$ time.

- We first compute the canonical representative of $c$ in $O(n + \ell)$ time.
- We compute $i(c)$ in the surface with all its faces perforated in $O(\ell^2)$ time.
- The minimum $i(c)$ is attained by some orders on the edges. It has no bigon.
- The corresponding representative has no non-flat bigon, no monogon and no singular flat bigon, so by Hass and Scott it is optimal.
System of Curves

Hass and Scott for singular bigons does not hold.

The computation of the minimal representative for a single curve cannot be extended to two curves.

However, Cohen and Lustig still works.

Although there might be no singular bigon, we have:

Lemma

If \((c, d)\) have excess intersections then there is a bigon between \(c\) and \(d\) (not necessarily singular).
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**Summary**

$k = \text{number of curves}$  
$b = \text{number of boundaries}$

<table>
<thead>
<tr>
<th>$k = 1, b &gt; 0$</th>
<th>Simple</th>
<th>Number</th>
<th>Representative</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$O((g\ell)^2)$</td>
<td>$O((g\ell)^2)$</td>
<td>$O((g\ell)^4)$</td>
</tr>
</tbody>
</table>

| $k = 2, b > 0$   | n.a.            | $O((g\ell)^2)$ | ?              |

| $k = 1, b = 0$   | ?               | ?               | ?              |

| $k = 2, b = 0$   | n.a.            | ?               | ?              |

<table>
<thead>
<tr>
<th>$k = 1$</th>
<th>Simple</th>
<th>Number</th>
<th>Representative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 1$</td>
<td>$O(\ell \cdot \log^2(\ell))$</td>
<td>$O(\ell^2)$</td>
<td>$O(\ell^4)$</td>
</tr>
<tr>
<td>$k = 2$</td>
<td>n.a.</td>
<td>$O(\ell^2)$</td>
<td>?</td>
</tr>
</tbody>
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Thank you
Easy Cases