Simple cycles, the case of splitting cycles

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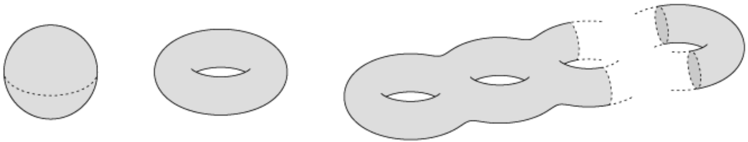
Surfaces

Definition

A surface is a compact, connected 2-manifold without boundary.



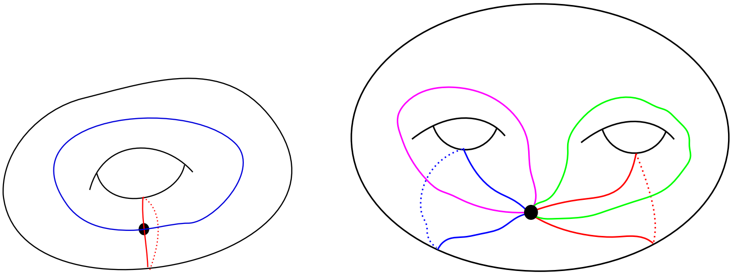
Genus

The genus of a surface is its number of handles.

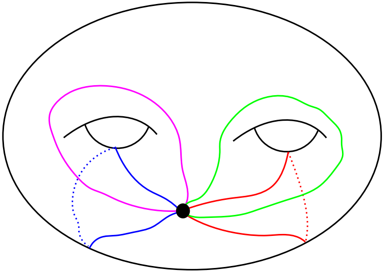
Free homotopy

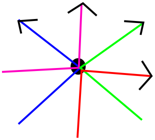
Two curves c_1 and c_2 are freely homotopic iff c_1 can be continously deform on c_2.

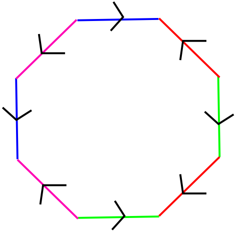
We fix a base point x_0 on the surface S. The corresponding homotopy group is called $\pi_1(S)$.

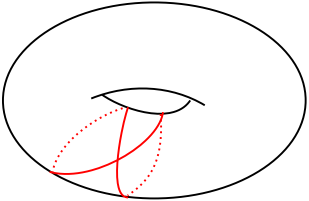


Vincent DESPRE









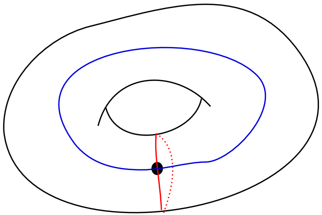
Simple cycle

A cycle on S is simple iff it has no double points.

Embedded graphs

Embedding

An embedding of a graph G on a surface S is a proper drawing of G on S.



Cellular embedding

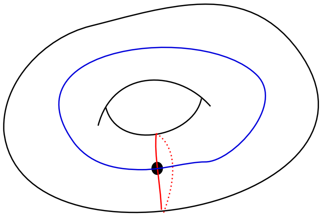
An embedding is a cellular embedding if the surface after cutting along the graph is an union of open disks.

Euler Characteristic

Definition

let S be a surface of genus g and G a graph cellularly embedded on S with v vertices, e edges and f faces, then:

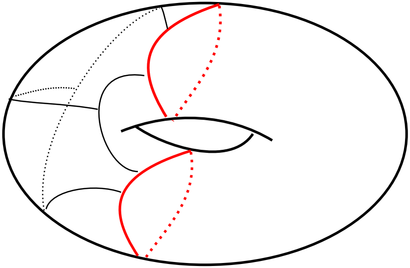
$$\chi(S) = v - e + f = 2 - 2g$$



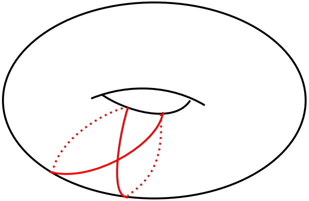
$$\chi(S) = 1 - 2 + 1 = 2 - 2 * 1 = 0$$

Homologous cycles

Two cycles on the graph C_1 and C_2 are homologous iff there is a set a faces such that the boundary of the union of them are exactly $C_1 \cup C_2$.



The homology group is called $H_1(S)$.



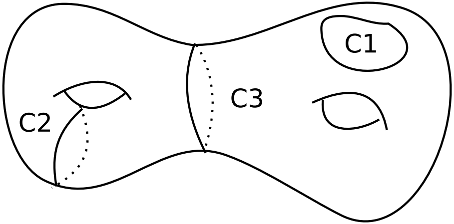
Simple cycle

A cycle of G is simple iff it has no repeated vertex.

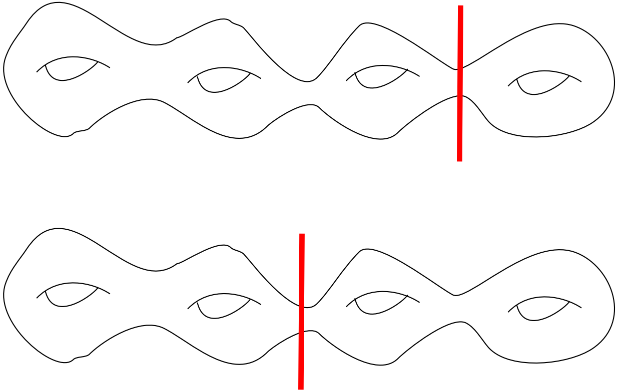
Splitting cycle

3 kinds of cycles:

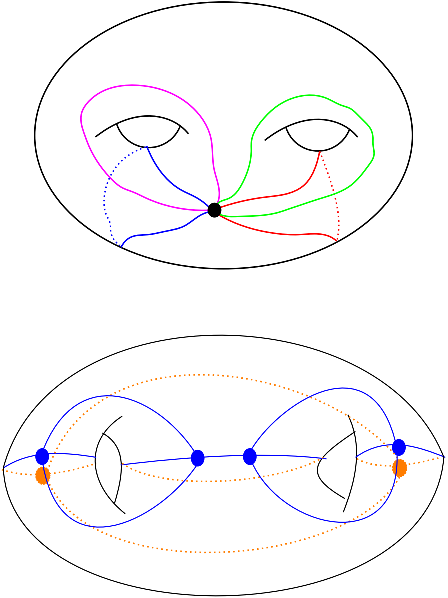
- Contractible and separating cycles (C1).
- Non-contractible and non-separating cycles (C2).
- Non-contractible and separating cycles also called **splitting cycles** (C3).



Different cuttings



Examples



Status of the problem

Complexity (Colin de Verdière et al. 2008)

Deciding if a combinatorial surface has a splitting cycle is NP-complete.

Conjecture (Barnette, 1982)

Every triangulation of a surface of genus at least 2 has a splitting cycle.

Conjecture (Mohar et Thomassen, 2001)

For all triangulation S of genus g and all $h \in [\![1, g-1]\!]$, there is a splitting cycle that separates S into two pieces of genera h and $g-h$.

Results

Conjecture (Mohar et Thomassen, 2001)

For all triangulation S of genus g and all $h \in [\![1, g-1]\!]$, there is a splitting cycle that separates S into two pieces of genera h and $g-h$.

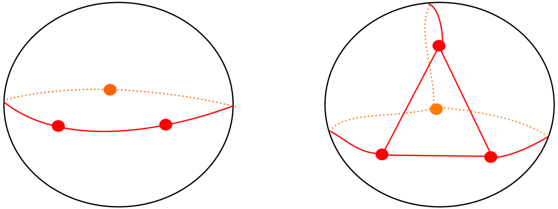
Counter-example

There is a triangulation of genus 20 without splitting cycles for $h \in \{5, \cdots, 10\}$.

Triangulations

Definition

A triangulation of a surface S is a simplicial complex C and a homeomorphism between S and C.

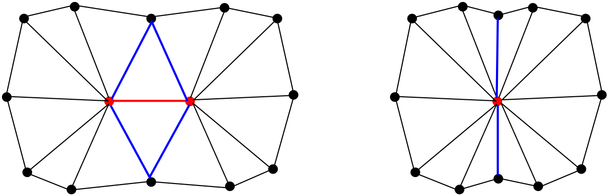


This definition excludes:

- Loop edges.
- 2 edges with the same end points.
- 2 faces that share their 3 vertices.

Irreducible triangulations

A triangulation is irreducible if none of its edges can be contracted.



For a fixed genus g, there is a finite number of irreducible triangulations with bounded number of vertices (less than $13g - 4$ [Joret and Wood, 2010]) .

Consequence

The conjecture is decidable for fixed genus.

Genus 2 irreducible triangulations

First implementation by Thom Sulanke.

Genus 2:

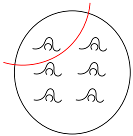
Number of triangulations: 396 785

•	3	4	5	6	7	8	Average
10		2	51	681	130	1	6.09
11	2	58	2249	16138	7818	11	6.21
12	25	1516	20507	72001	22877	121	6.00
13	710	13004	50814	78059	16609	9	5.61
14	8130	30555	12308	3328	205	1	4.21
15	36794	1395	3	1	2		3.04
16	661	3					3.01
17	5						3.00

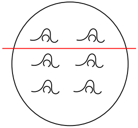
Genus 6

We consider the 59 non-isomorphic embeddings of K_{12}.

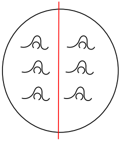
Average: 7.58
Worst-case: 8



Average: 9.41
Worst-case: 10

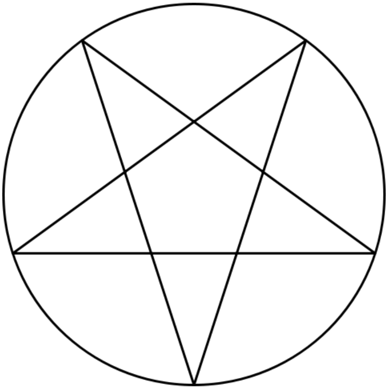


Average: 10.32
Worst-case: 12 (Hamiltonian cycle!)



Idea, case of K_5

K_n has $\binom{n}{2}$ edges.
Example of K_5:



Number of edges: $\binom{5}{2} = 10$
Number of faces: $3e = 2f \Rightarrow f = \frac{2}{3}e = \frac{20}{3} \notin \mathbb{N}$

General case

$$\chi(S) = v - e + f = n - \frac{n(n-1)}{2} + \frac{2}{3} \cdot \frac{n(n-1)}{2} = 2 - 2g$$

$$g = \frac{(n-3)(n-4)}{12}$$

$$(n-3)(n-4) \equiv 0[12] \Leftrightarrow n \equiv 0, 3, 4 \text{ or } 7[12]$$

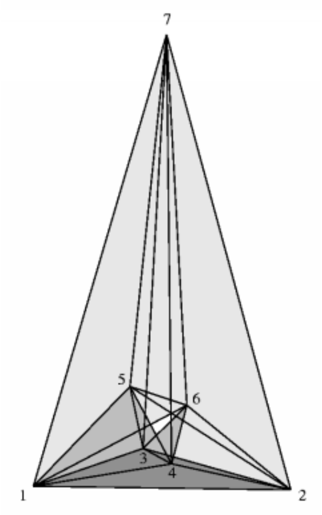
Theorem (Ringel and Youngs, ∼1970)

K_n can triangulate a surface if and only if $n \equiv 0, 3, 4$ or $7[12]$.

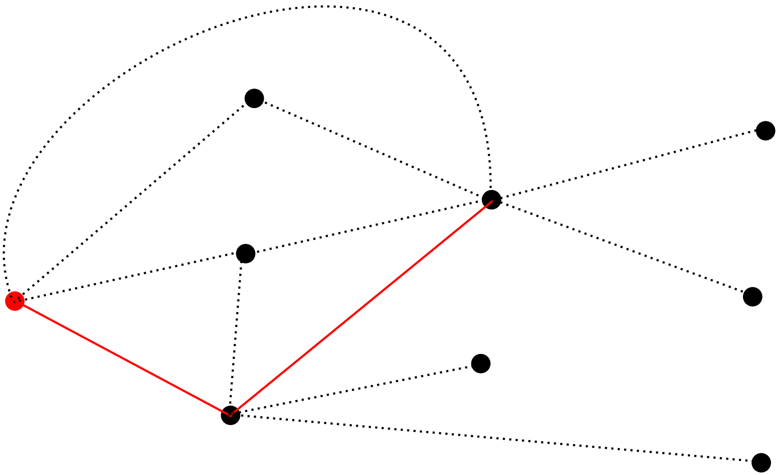
K_7

"K_7 embedded on a torus," by sarah-marie belcastro (Hadley, MA)

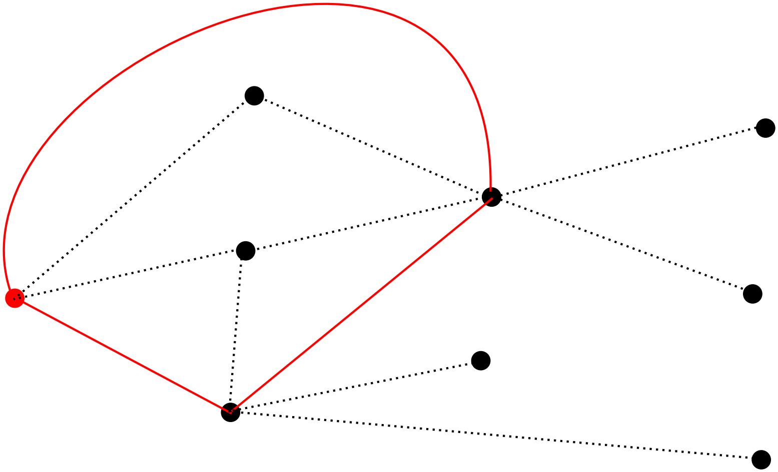
K_7



Naïve algorithm



Naïve algorithm



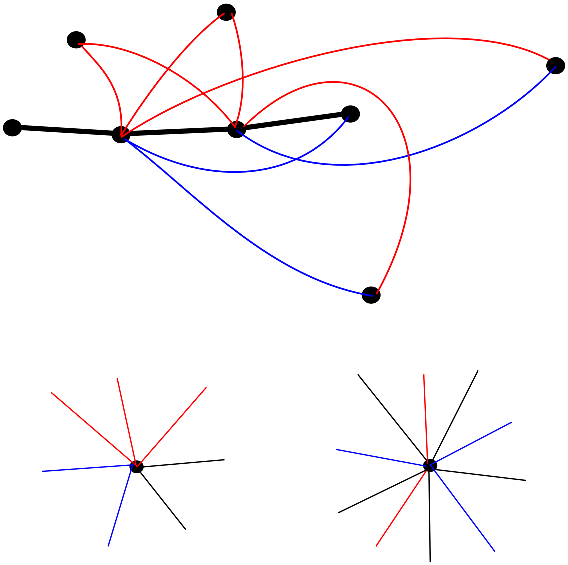
Computation time

New implementation in C++. The data-structure used for the triangulations is the flag representation (a.k.a. winged-edges).

n	12	15	16	19	31	43
basic	2 s.	1 h.	12 h.	$\sim$10 years		
final	10 s.	20 s.	25 s.	1 m.	25 m.	10 h.

This has been computed with an 8 cores computer with 16 Go of RAM. It uses parallel computation.

An edge-coloring algorithm



Results, case of K_{19}

Genera	Shortest splitting cycle	#cycles
$1 \to 19$	10	2080
$2 \to 18$	14	1374
$3 \to 17$	18	278
$4 \to 16$	19	38
$5 \to 15$	$\perp$	0
$6 \to 14$	$\perp$	0
$7 \to 13$	$\perp$	0
$8 \to 12$	$\perp$	0
$9 \to 11$	$\perp$	0
$10 \to 10$	$\perp$	0

•	15	19	27	28	31	39	40	43
1	8	10	12	12	8	12	10	8
2	11	14	16	17	13	15	15	11
3	12	18	19	18	17	20	18	12
4	13	19	20	$\perp$	19	24	19	15
5	14	$\perp$	27	$\perp$	20	26	24	18
6		$\perp$	$\perp$	$\perp$	25	30	26	23
7		$\perp$	$\perp$	$\perp$	26	32	28	26
8		$\perp$	$\perp$	$\perp$	26	$\perp$	30	26
9		$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	33	28
10		$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	35	34
11			$\perp$	$\perp$	$\perp$	$\perp$	36	34
12			$\perp$	$\perp$	$\perp$	$\perp$	38	35
13			$\perp$	$\perp$	$\perp$	$\perp$	40	35
14			$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

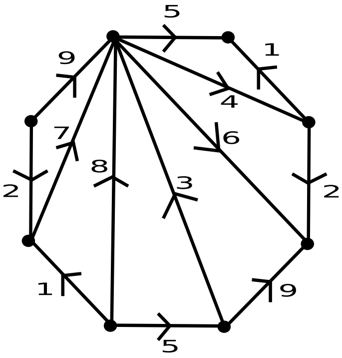
K_{31}: only 78 of the 130 980 splitting cycles are separating the surface into 2 pieces of genera 8 and 55.

Known results

We talk about K_n with $n = 12 \cdot s + i$.

- i = 0, 3, 4 and 7, there is an embedding (Ringel and Youngs 1972).
- i = 3, 4 and 7, there are O(4^s) non-isomorphic embeddings (Korzhik and Voss 2001).
- n = 12 (NO), there are exactly 182 200 non-isomorphic embeddings (Ellingham and Stephen 2003).
- n = 13 (NO), there are exactly 243 088 286 non-isomorphic embeddings (Ellingham and Stephen 2003).
- i = 3 and 7 and for an infinite numbers of values of n, there are at least $n^{c \cdot n^2}$ 2-colorable non-isomorphic embeddings (Granell and Knor 2012).

Example $n \equiv 7$



Rotation scheme of 0:

$$0 : (9, 7, 8, 3, 13, 15, 14, 11, 18, 4, 17, 10, 16, 5, 1, 12, 2, 6)$$

The other rotation scheme comes from the addition in $\mathbb{Z}/19\mathbb{Z}$:

$$1 : (10, 8, 9, 4, 14, 16, 15, 12, 0, 5, 18, 11, 17, 6, 2, 13, 3, 7)$$

Face-width

Let G be a graph with a 2-cell embedding on a surface S.

Definition (Face-width)

The **face-width** of G is the minimum number of intersection between any non-contractile cycle of S and the graph G.

Theorem

Let f a face, $k \in \mathbb{N}^*$ and $E_{f,k} = \{f', d(f, f') \leq k\}$. If $fw(G) > 2k + 1$ then there is a disk D such that:

$$E_{f,k} \subset D \ \ and \ \ \partial D \subset \partial E_{f,k}$$

Face-width

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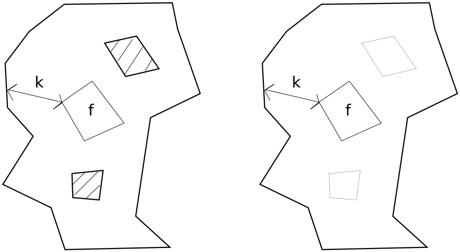


Conjectures

Conjecture (Zha, 1991)

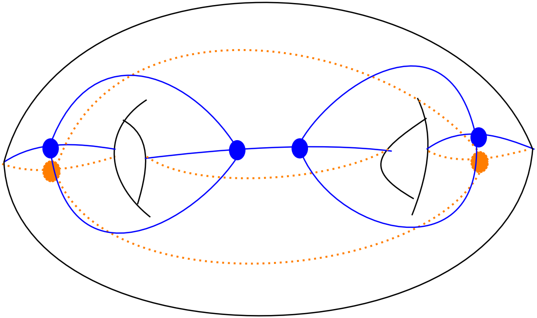
Every combinatorial map of genus at least 2 and face-width at least 3 has a splitting cycle.

Conjecture (Zha, 1991)

Every combinatorial map of genus $g \geq 2$ and face-width at least 3 has a splitting cycle that split the surface into one surface of genus h and one of genus $g-h$, for all $0 < h < g$.

Optimality

A combinatorial map of genus 2 and face-width 2 without splitting cycle:



Results

In the orientable case:

- The conjecture is true for face-width at least 6 (Zha and Zhao 1993).
- The conjecture is true for genus 2 and face-width at least 4 (Ellingham and Zha 2003).

In the non-orientable case:

- The conjecture is true for face-width at least 5 (Zha and Zhao 1993).
- The conjecture is true for genus 2 (Robertson and Thomas 1991).

Perspectives

- Give a formal proof (without computer) that Ringel's embedding of K_{19} do not have a splitting cycle that separates the surfaces into 2 pieces of genera 10.
- Analyze the complexity of the edge-coloring algorithm in the case of complete graphs.
- Prove the existence of splitting cycles of type 1 and $g-1$ in every triangulation.

This is the end