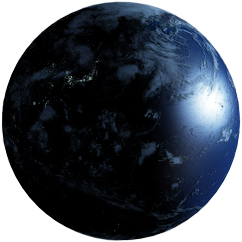
PhD Defense

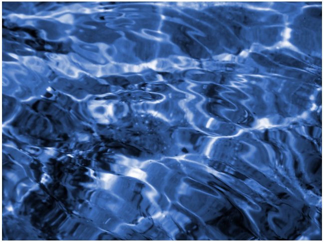
Topology and Algorithms on Combinatorial Maps

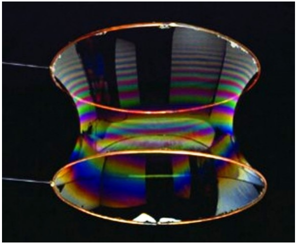
Vincent Despré

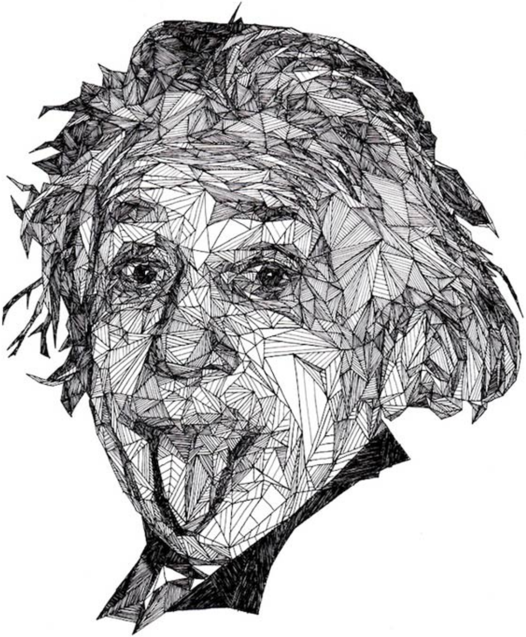
Gipsa-lab, G-scop, Labex Persyval

18 Octobre 2016







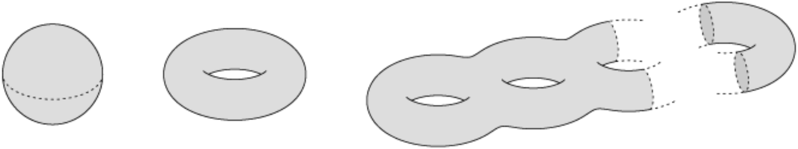


V=number of vertices, E=number of edges and F=number of faces

Euler Formula

On a surface that can be deformed to a sphere, any polygonal subdivision verifies:

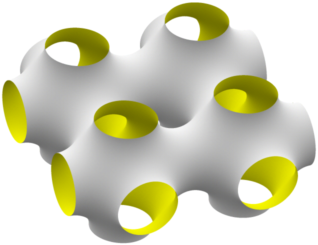
$$\chi(S) = V - E + F = 2$$



Euler Formula

On a surface S of genus g, any polygonal subdivision verifies:

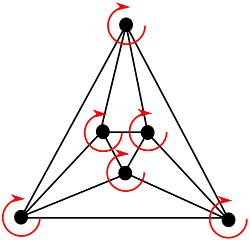
$$\chi(S) = V - E + F = 2 - 2g$$

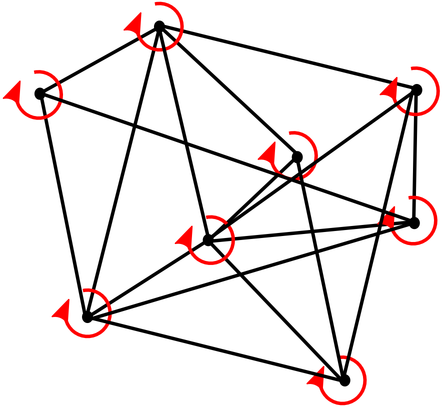


Euler Formula

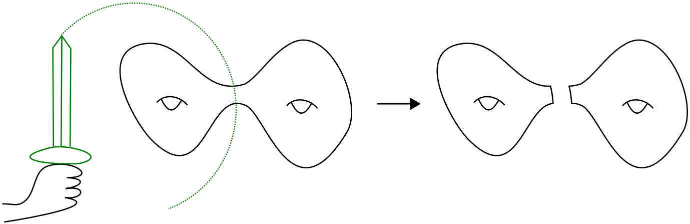
On a surface S of genus g with b boundaries, any polygonal subdivision verifies:

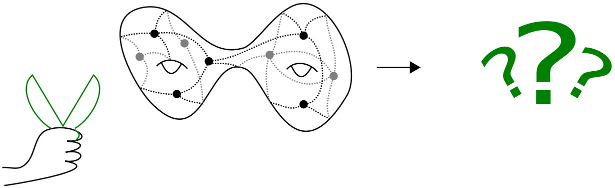
$$\chi(S) = V - E + F = 2 - 2g - b$$



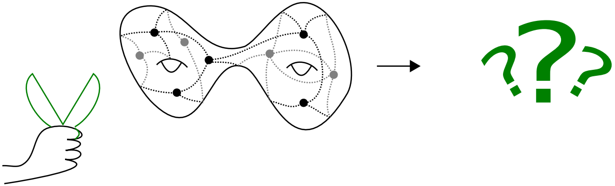


Splitting Cycles





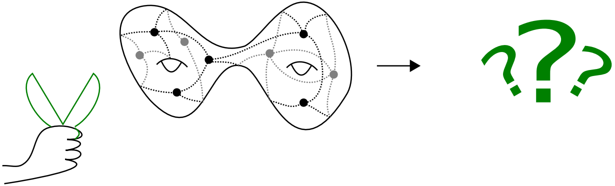
Splitting Cycles



Cabello et al. (2011)

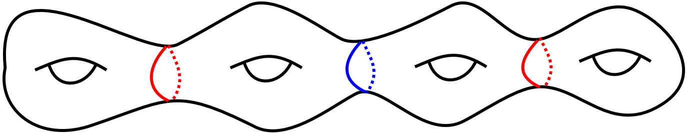
Deciding if a combiantorial map admits a splitting cycle is NP-complete.

Splitting Cycles



Barnette's Conjecture (1982)

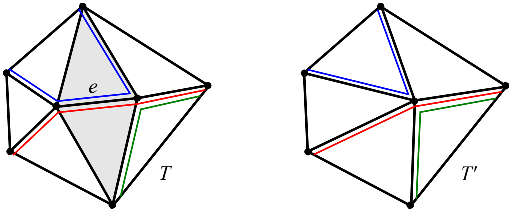
Every triangulations of surfaces of genus at least 2 admit a splitting cycle.



Conjecture (Mohar and Thomassen, 2001)

Every triangulations of surfaces of genus $g \geq 2$ admit a splitting cycle of every different type.

Irreducible Triangulations



- There are a finite number of irreducible triangulations of genus g. (Barnette and Edelson, 1988 and Joret and Wood, 2010)
- There are 396784 irreducible triangulations of genus 2.
- Unreachable for genus 3.

Genus 2 irreducible triangulations

First implementation by Thom Sulanke.

Genus 2:

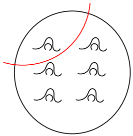
Number of triangulations: 396 784

n \ l	3	4	5	6	7	8	Average
10		2	51	681	130	1	6.09
11	2	58	2249	16138	7818	11	6.21
12	25	1516	20507	72001	22877	121	6.00
13	710	13004	50814	78059	16609	9	5.61
14	8130	30555	12308	3328	205	1	4.21
15	36794	1395	3	1	2		3.04
16	661	3					3.01
17	5						3.00

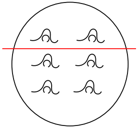
Genus 6

We consider the 59 non-isomorphic embeddings of K_{12}. (Altshuler, Bokowski and Schuchert 1996)

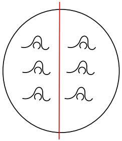
Average: 7.58
Worst-case: 8



Average: 9.41
Worst-case: 10



Average: 10.32
Worst-case: 12 (Hamiltonian cycle!)



Complete Graphs

$$\chi(S) = v - e + f = n - \frac{n(n-1)}{2} + \frac{2}{3} \cdot \frac{n(n-1)}{2} = 2 - 2g$$

$$g = \frac{(n-3)(n-4)}{12}$$

$$(n-3)(n-4) \equiv 0[12] \Leftrightarrow n \equiv 0, 3, 4 \text{ or } 7[12]$$

Theorem (Ringel and Youngs, ~1970)

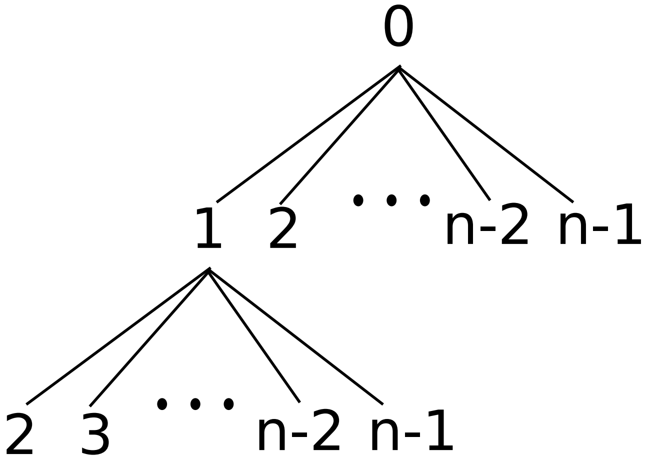
K_n can triangulate a surface if and only if $n \equiv 0, 3, 4$ or $7[12]$.

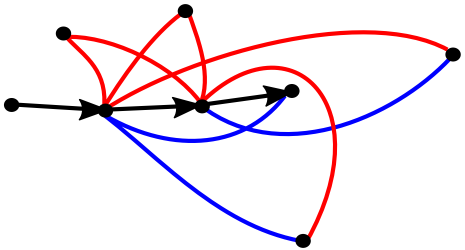
Computation time

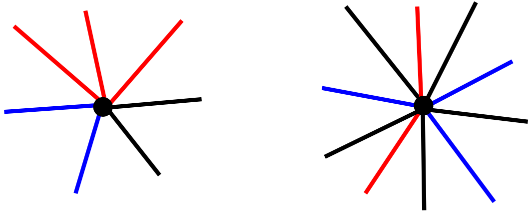
New implementation in C++. The data-structure used for the triangulations is the flag representation.

n	12	15	16	19
basic	2 s.	1 h.	12 h.	$\sim$10 years

This has been computed with an 8 cores computer with 16 Go of RAM. It uses parallel computation.


0
1
2
...
n-2
n-1
2
3
...
n-2
n-1





n	15	16	19	···	43
basic	1 h.	12 h.	∼10 years		
final	2 s.	3 s.	8 sec.		1 h.

Type \ K_n	K_{15}	K_{16}	K_{19}	K_{27}	K_{28}	K_{31}	K_{39}	K_{40}	K_{43}
1	8	10	11	12	12	8	12	10	8
2	11	12	14	16	17	13	15	15	11
3	12	14	16	19	18	15	20	18	12
4	13	16	18	20	$\perp$	17	24	19	15
5	14	16	$\perp$	27	$\perp$	20	26	24	18
6		16	$\perp$	$\perp$	$\perp$	21	30	26	20
7			$\perp$	$\perp$	$\perp$	23	32	28	21
8			$\perp$	$\perp$	$\perp$	24	$\perp$	30	23
9			$\perp$	$\perp$	$\perp$	28	$\perp$	33	24
10			$\perp$	$\perp$	$\perp$	28	$\perp$	35	25
11				$\perp$	$\perp$	29	$\perp$	36	27
12				$\perp$	$\perp$	$\perp$	$\perp$	38	29
13				$\perp$	$\perp$	$\perp$	$\perp$	40	30
14				$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	31
⋮				$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	⋮
29						$\perp$	$\perp$	$\perp$	42
30						$\perp$	$\perp$	$\perp$	$\perp$
max type	5	6	10	23	25	31	52	55	65

$\perp$ = No cycle found.

Counter-Examples

Mohar and Thomassen conjecture is false.

Type \ K_n	K_{15}	K_{16}	K_{19}	K_{27}	K_{28}	K_{31}	K_{39}	K_{40}	K_{43}
1	8	10	11	12	12	8	12	10	8
2	11	12	14	16	17	13	15	15	11
3	12	14	16	19	18	15	20	18	12
4	13	16	18	20	$\perp$	17	24	19	15
5	14	16	$\perp$	27	$\perp$	20	26	24	18
6		16	$\perp$	$\perp$	$\perp$	21	30	26	20
7			$\perp$	$\perp$	$\perp$	23	32	28	21
8			$\perp$	$\perp$	$\perp$	24	$\perp$	30	23
9			$\perp$	$\perp$	$\perp$	28	$\perp$	33	24
10			$\perp$	$\perp$	$\perp$	28	$\perp$	35	25
11				$\perp$	$\perp$	29	$\perp$	36	27
12				$\perp$	$\perp$	$\perp$	$\perp$	38	29
13				$\perp$	$\perp$	$\perp$	$\perp$	40	30
14				$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	31
⋮				$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	⋮
29						$\perp$	$\perp$	$\perp$	42
30						$\perp$	$\perp$	$\perp$	$\perp$
max type	5	6	10	23	25	31	52	55	65

$\perp$ = No cycle found.

Conjecture

For every $\alpha > 0$, there exists a triangulation with no splitting cycles of type larger than $\alpha \cdot \frac{g}{2}$.

Encoding Toroidal Triangulations

Properties of the planar case:

1/ We have a notion of 3-orientation for triangulations.

2/ Every 3-orientation admits a unique Schnyder wood coloration.

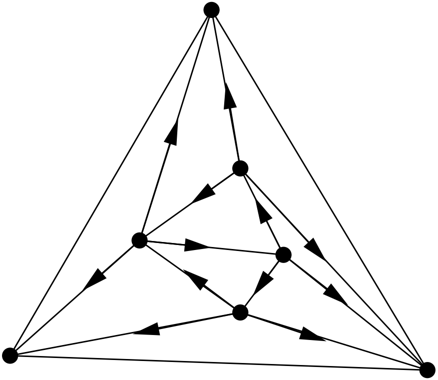
3/ Each color corresponds to a spanning tree and so there is no monochromatic cycle.

4/ The 3-orientations of a given triangulation have a structure of distributive lattice.

5/ The minimal element of the lattice has no clockwise oriented cycle.

6/ Triangulations are in bijection with a particular type of decorated embedded trees.

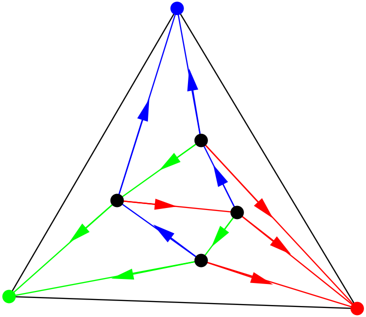
1/ We have a notion of 3-orientation for triangulations.

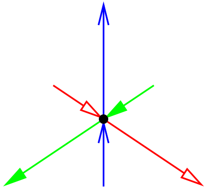


Kampen (1976)

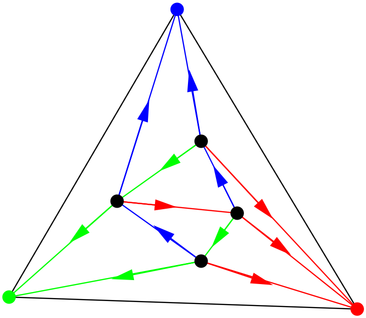
Every planar triangulation admits a 3-orientation.

2/ Every 3-orientation admits a unique Schnyder wood coloration.





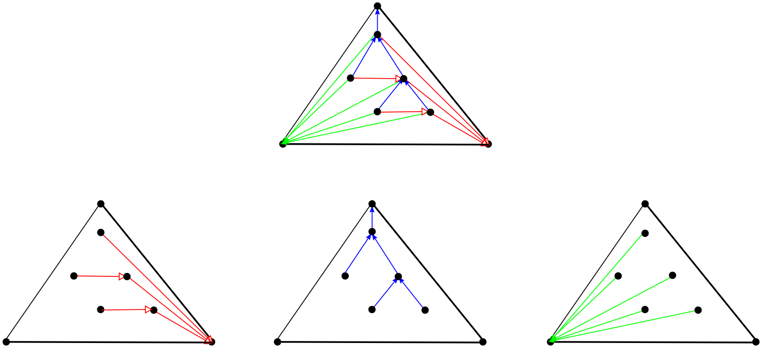
2/ Every 3-orientation admits a unique Schnyder wood coloration.



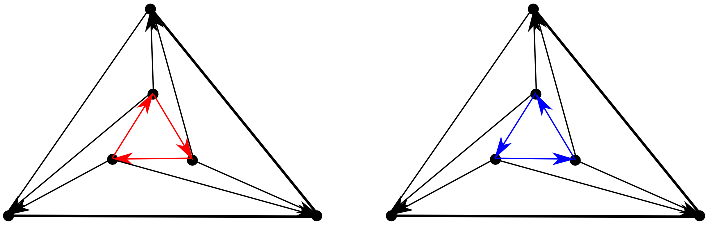
de Fraisseix and Ossona de Mendez (2001)

Each 3-orientation of a plane simple triangulation admits a unique coloring (up to permutation of the colors) leading to a Schnyder wood.

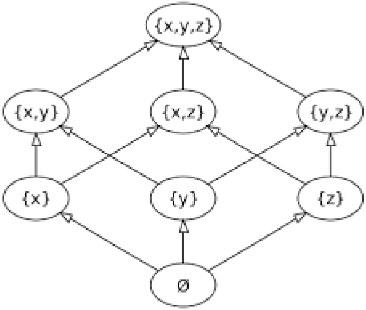
3/ Each color corresponds to a spanning tree and so there is no monochromatic cycle.



4/ The 3-orientations of a given triangulation have a structure of distributive lattice.



4/ The 3-orientations of a given triangulation have a structure of distributive lattice.



Propp (1993), Ossona de Mendez (1994), Felsner (2004)

The set of the 3-orientations of a given triangulation has a structure of distributive lattice for the appropriate ordering.

Properties of the planar case:

1/ We have a notion of 3-orientation for triangulations.

2/ Every 3-orientation admits a unique Schnyder wood coloration.

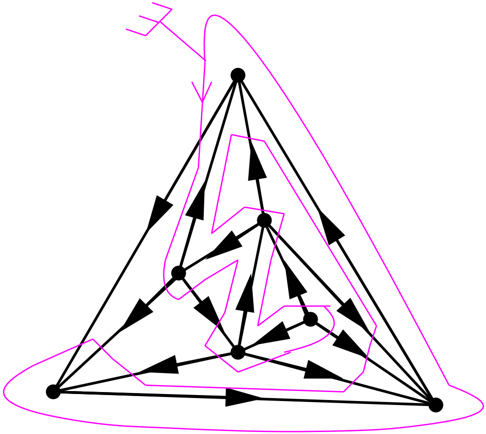
3/ Each color corresponds to a spanning tree and so there is no monochromatic cycle.

4/ The 3-orientations of a given triangulation have a structure of distributive lattice.

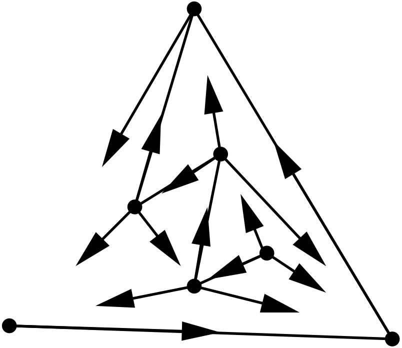
5/ The minimal element of the lattice has no clockwise oriented cycle.

6/ Triangulations are in bijection with a particular type of decorated embedded trees.

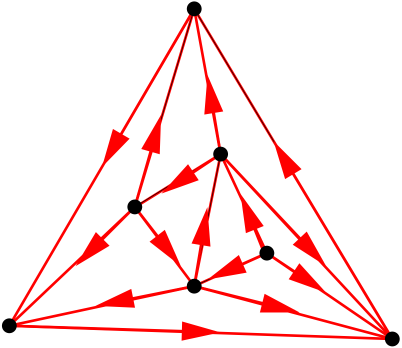
6/ Triangulations are in bijection with a particular type of decorated embedded trees **(Poulalhon and Schaeffer, 2006)**.



6/ Triangulations are in bijection with a particular type of decorated embedded trees **(Poulalhon and Schaeffer, 2006)**.



6/ Triangulations are in bijection with a particular type of decorated embedded trees **(Poulalhon and Schaeffer, 2006)**.



Properties of the torus case:

1/ We have a notion of 3-orientation for triangulations.

2/ Every 3-orientation admits a unique Schnyder wood coloration.

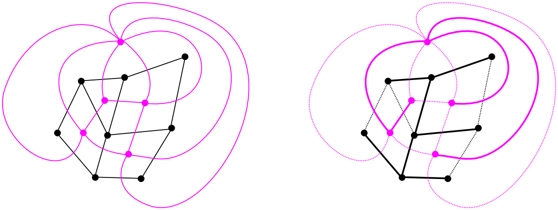
3/ ~~Each color corresponds to a spanning tree and so~~ There is no monochromatic **contractible** cycle.

4/ The 3-orientations of a given triangulation have a structure of distributive lattice**s**.

5/ The minimal element of each lattice has no clockwise oriented contractible cycle.

6/ Triangulations are in bijection with a particular type of decorated **unicellular toroidal maps**.

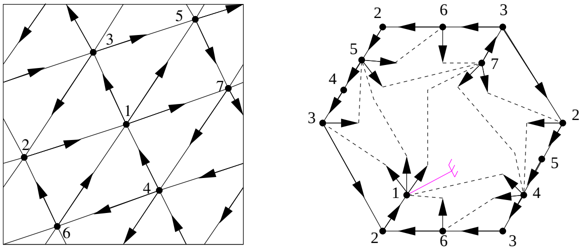
6/ Triangulations are in bijection with a particular type of decorated **unicellular toroidal maps**.



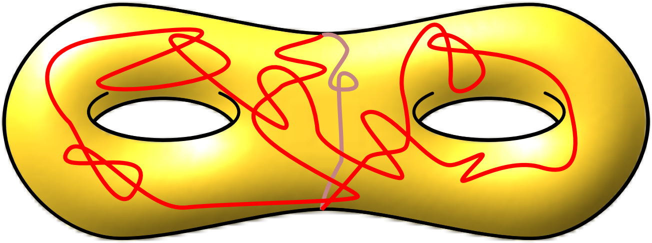
Tree-cotree Decomposition: (T, C, X). T has $n-1$ edges, C has $f-1$ edges and X the remaining.

$\chi = n - (n - 1 + f - 1 + x) + f \Leftrightarrow x = 2 - \chi = 2g$

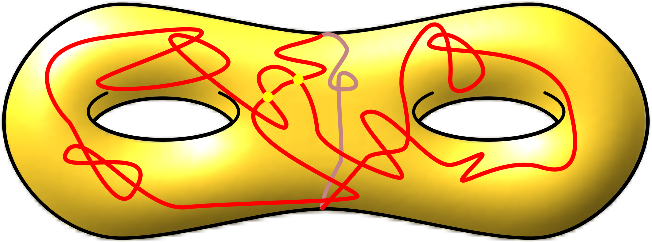
6/ Triangulations are in bijection with a particular type of decorated **unicellular toroidal maps**.



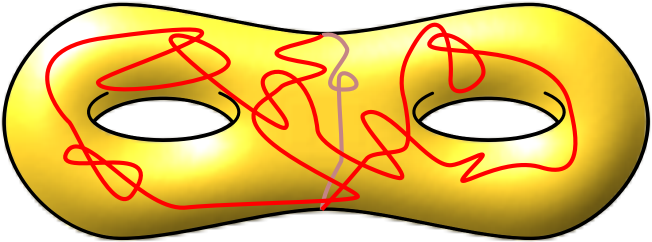
Geometric Intersection Number of Curves



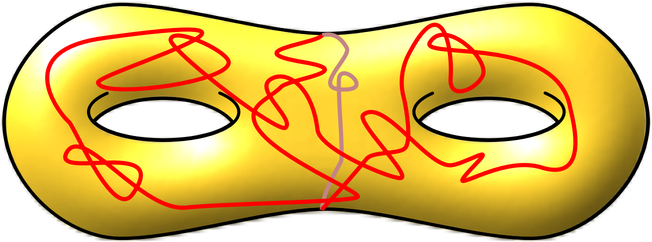
Geometric Intersection Number of Curves



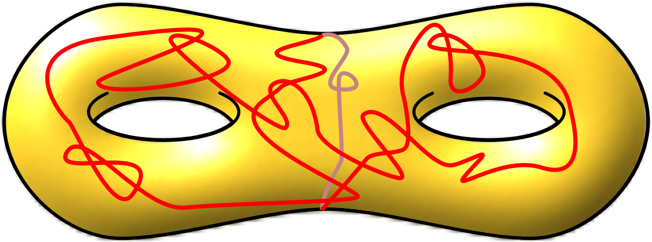
Geometric Intersection Number of Curves



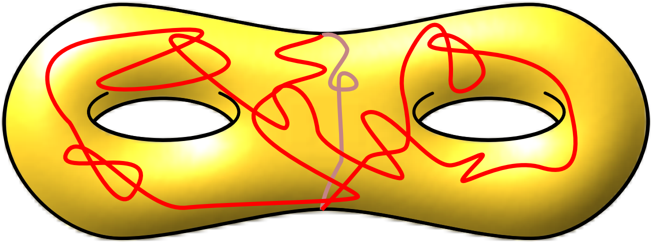
Geometric Intersection Number of Curves



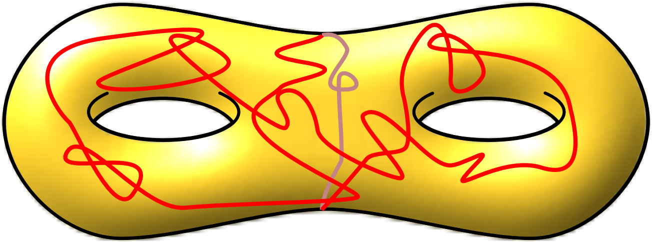
Geometric Intersection Number of Curves



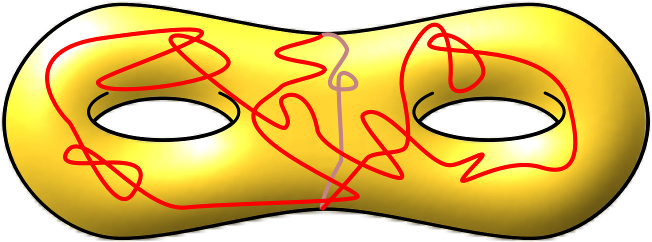
Geometric Intersection Number of Curves



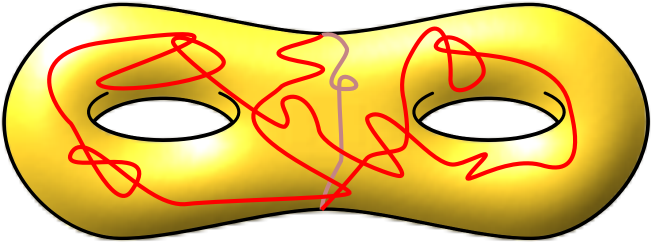
Geometric Intersection Number of Curves



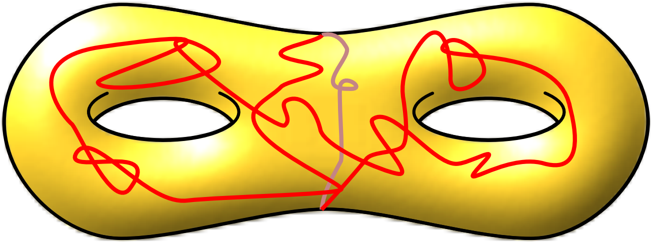
Geometric Intersection Number of Curves



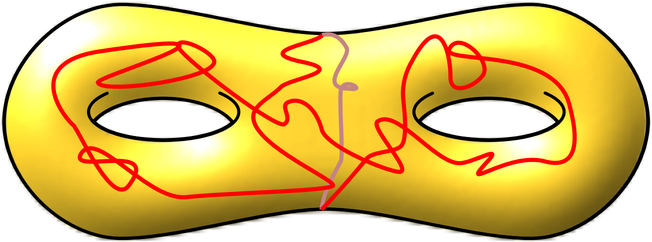
Geometric Intersection Number of Curves



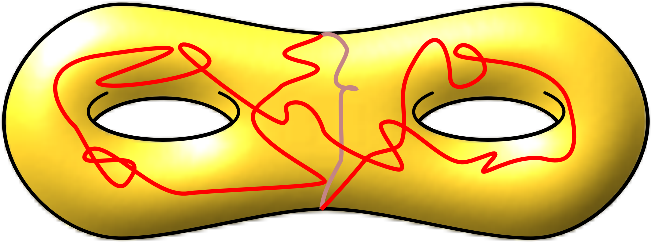
Geometric Intersection Number of Curves



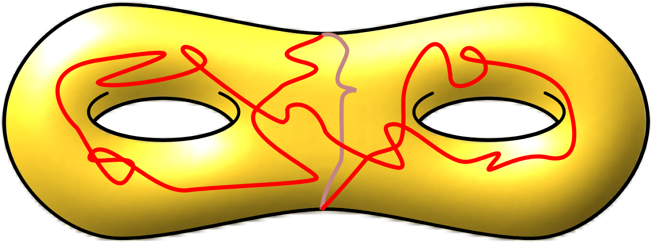
Geometric Intersection Number of Curves



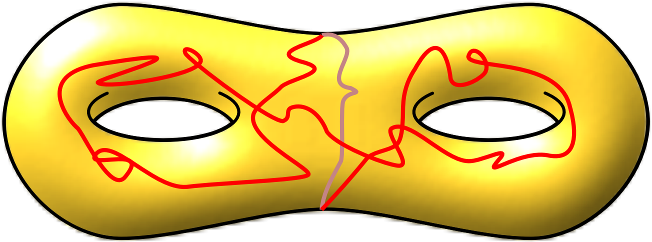
Geometric Intersection Number of Curves



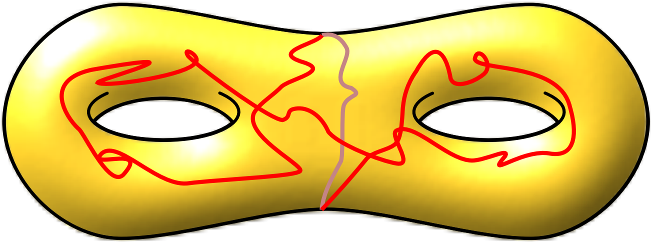
Geometric Intersection Number of Curves



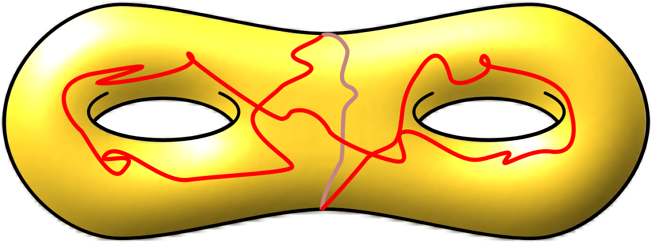
Geometric Intersection Number of Curves



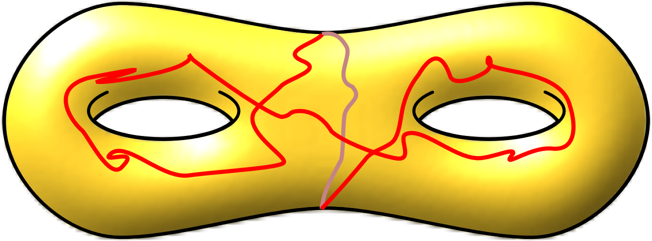
Geometric Intersection Number of Curves



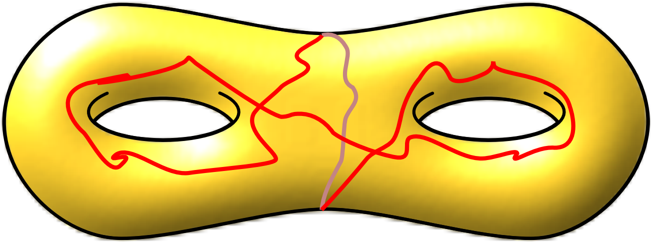
Geometric Intersection Number of Curves



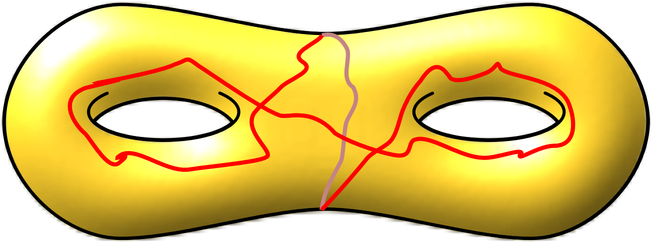
Geometric Intersection Number of Curves



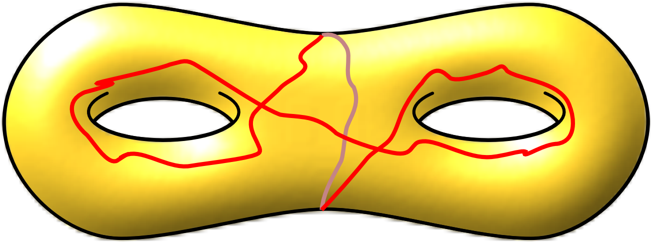
Geometric Intersection Number of Curves



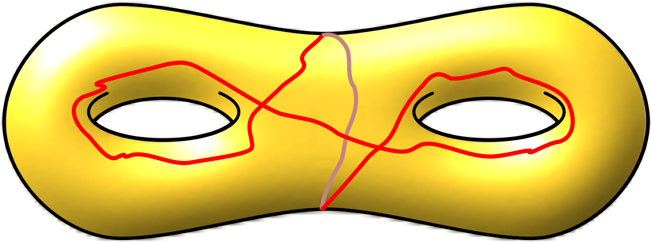
Geometric Intersection Number of Curves



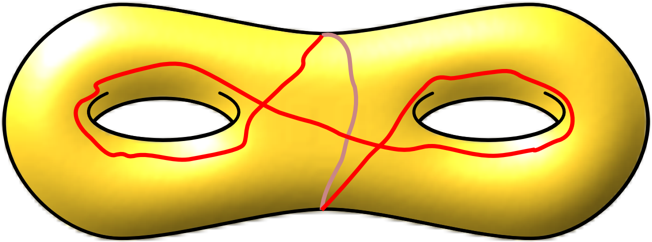
Geometric Intersection Number of Curves



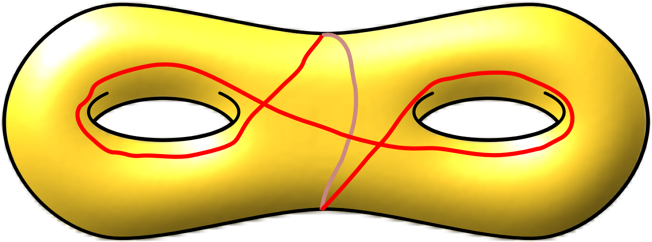
Geometric Intersection Number of Curves



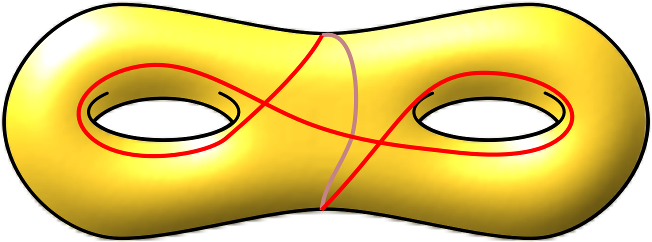
Geometric Intersection Number of Curves



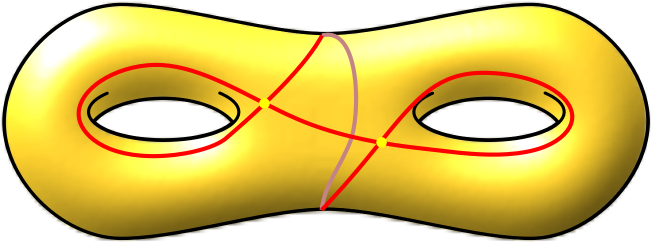
Geometric Intersection Number of Curves

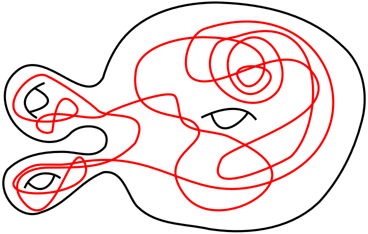


Geometric Intersection Number of Curves

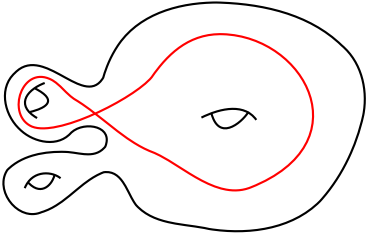


Geometric Intersection Number of Curves





(g) Number of crossings: too many!



(h) Number of crossings: 1 $\rightarrow$ optimal

Three problems:

- Deciding if a curve can be made simple by homotopy.
- Finding the minimum possible number of self-intersections.
- Finding a corresponding minimal representative.

Boundaries	Simple	Number	Representative
$b > 0$	$O((g\ell)^2)$ BS (1984)	$O((g\ell)^2)$ CL (1987)	$O((g\ell)^4)$ A (2015)
$b = 0$	? L (1987) $O(\ell^5)$ GKZ (2005)	? L (1987) $O(\ell^5)$ GKZ (2005)	? dGS (1997)
Any	$O(\ell \cdot \log^2(\ell))$	$O(\ell^2)$	$O(\ell^4)$

BS: Birman and Series, An algorithm for simple curves on surfaces.
CL: Cohen and Lustig, Paths of geodesics and geometric intersection numbers: I.
L: Lustig, Paths of geodesics and geometric intersection numbers: II.
A: Arettines, A combinatorial algorithm for visualizing representatives with minimal self-intersection.
dGS: de Graaf and Schrijver, Making curves minimally crossing by Reidemeister moves.
GKZ: Gonçalves, Kudryavtseva and Zieschang, An algorithm for minimal number of (self-)intersection points of curves on surfaces.

Publications:

1/ Some Triangulated Surfaces without Balanced Splitting: Published in *Graphs and Combinatorics*.

2/ Encoding Toroidal Triangulations: Accepted in *Discrete & Computationnal Geometry*.

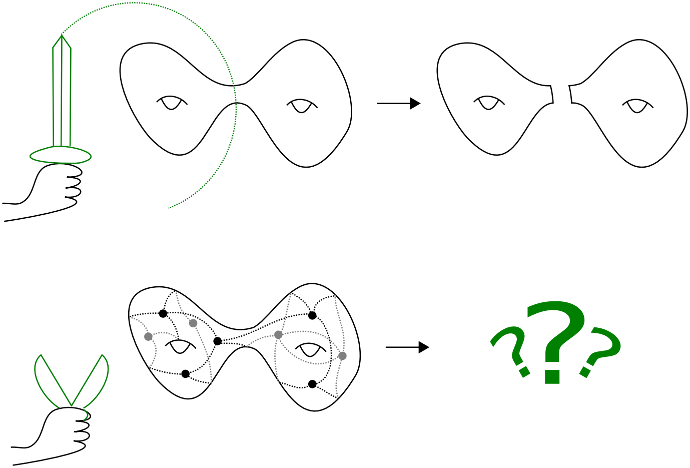
3/ Computing the Geometric Intersection Number of Curves: Will be submitted to the next SoCG.

Work in progress:

1/ Looking for a proof that does not require a computer.

2/ There are a lot of implications for the bijection in the plane. Is it possible to generalized them.

3/ It remains to look at the construction of a minimal representative for a couple of curves.



Conjecture

Deciding if there is a simple closed walk in a given homotopy class is NP-complete and FPT parametrized by the genus of the surface.

Do you have questions?