Topology and Algorithms on Combinatorial Maps

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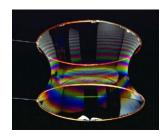
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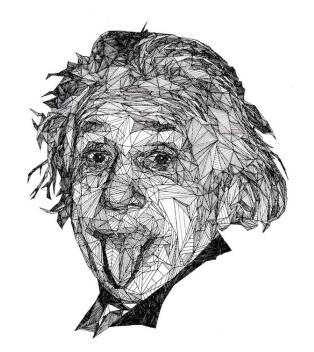
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Topology

Number of



V=number of vertices, E=number of edges and F=number of faces

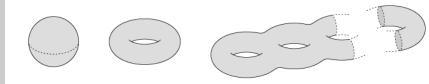
Euler Formula

On a surface that can be deformed to a sphere, any polygonal subdivision verifies:

$$\chi(S) = V - E + F = 2$$

Topology

Number of



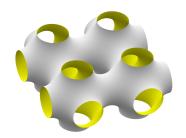
Euler Formula

On a surface S of genus g, any polygonal subdivision verifies:

$$\chi(S) = V - E + F = 2 - 2g$$

Topology

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Euler Formula

On a surface S of genus g with b boundaries, any polygonal subdivision verifies:

$$\chi(S) = V - E + F = 2 - 2g - b$$

Combinatorial Maps

Cycles

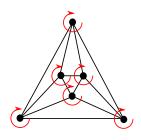
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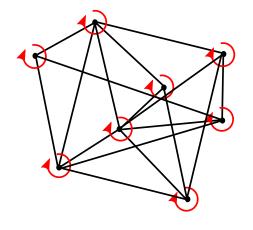
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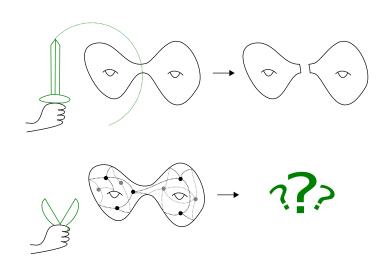
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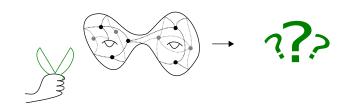
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Cabello et al. (2011)

Deciding if a combiantorial map admits a splitting cycle is NP-complete.

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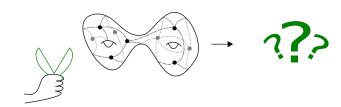
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Barnette's Conjecture (1982)

Every triangulations of surfaces of genus at least 2 admit a splitting cycle.

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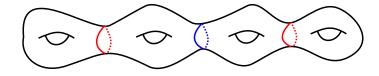
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Conjecture (Mohar and Thomassen, 2001)

Every triangulations of surfaces of genus $g \ge 2$ admit a splitting cycle of every different type.

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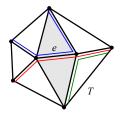
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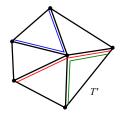
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Irreducible Triangulations





- → There are a finite number of irreducible triangulations of genus g. (Barnette and Edelson, 1988 and Joret and Wood, 2010)
- → There are 396784 irreducible triangulations of genus 2.
- → Unreachable for genus 3.

Genus 2 irreducible triangulations

First implementation by Thom Sulanke.

Genus 2:

Number of triangulations: 396 784

| n | 3 | 4 | 5 | 6 | 7 | 8 | Average |
|----|-------|-------|-------|-------|-------|-----|---------|
| 10 | | 2 | 51 | 681 | 130 | 1 | 6.09 |
| 11 | 2 | 58 | 2249 | 16138 | 7818 | 11 | 6.21 |
| 12 | 25 | 1516 | 20507 | 72001 | 22877 | 121 | 6.00 |
| 13 | 710 | 13004 | 50814 | 78059 | 16609 | 9 | 5.61 |
| 14 | 8130 | 30555 | 12308 | 3328 | 205 | 1 | 4.21 |
| 15 | 36794 | 1395 | 3 | 1 | 2 | | 3.04 |
| 16 | 661 | 3 | | | | | 3.01 |
| 17 | 5 | | | | | | 3.00 |

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Genus 6

We consider the 59 non-isomorphic embeddings of K_{12} . (Altshuler, Bokowski and Schuchert 1996)

Average: 7.58 Worst-case: 8

A A A

Average: 9.41 Worst-case: 10

A A A A

Average: 10.32

Worst-case: 12 (Hamiltonian cycle!)



Conclusion

Complete Graphs

$$\chi(S) = v - e + f = n - \frac{n(n-1)}{2} + \frac{2}{3} \cdot \frac{n(n-1)}{2} = 2 - 2g$$

$$g = \frac{(n-3)(n-4)}{12}$$

$$(n-3)(n-4) \equiv 0 [12] \Leftrightarrow n \equiv 0, 3, 4 \text{ or } 7[12]$$

Theorem (Ringel and Youngs, ∼1970)

 K_n can triangulate a surface if and only if $n \equiv 0, 3, 4$ or 7[12].

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Computation time

New implementation in C++. The data-structure used for the triangulations is the flag representation.

| n | 12 | 15 | 16 | 19 |
|-------|------|------|-------|-----------------|
| basic | 2 s. | 1 h. | 12 h. | \sim 10 years |

This has been computed with an 8 cores computer with 16 Go of RAM. It uses parallel computation.

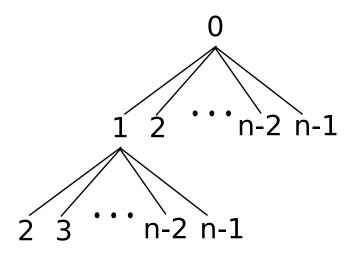
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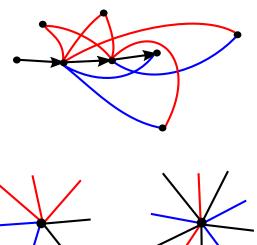
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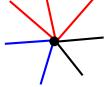
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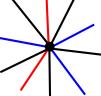
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| n | 15 | 16 | 19 | 43 |
|-------|------|-------|-----------------|--------|
| basic | 1 h. | 12 h. | \sim 10 years | |
| final | 2 s. | 3 s. | 8 sec. | 1 h. |

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| Type K_n | K_{15} | K_{16} | K_{19} | K_{27} | K_{28} | K_{31} | K_{39} | K_{40} | K_{43} |
|------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 1 | 8 | 10 | 11 | 12 | 12 | 8 | 12 | 10 | 8 |
| 2 | 11 | 12 | 14 | 16 | 17 | 13 | 15 | 15 | 11 |
| 3 | 12 | 14 | 16 | 19 | 18 | 15 | 20 | 18 | 12 |
| 4 | 13 | 16 | 18 | 20 | | 17 | 24 | 19 | 15 |
| 5 | 14 | 16 | | 27 | | 20 | 26 | 24 | 18 |
| 6 | | 16 | | | | 21 | 30 | 26 | 20 |
| 7 | | | | | | 23 | 32 | 28 | 21 |
| 8 | | | | | 1 | 24 | 1 | 30 | 23 |
| 9 | | | | | | 28 | | 33 | 24 |
| 10 | | | | | | 28 | | 35 | 25 |
| 11 | | | | | | 29 | | 36 | 27 |
| 12 | | | | | | | | 38 | 29 |
| 13 | | | | | | | | 40 | 30 |
| 14 | | | | | | T | | | 31 |
| | | | | | | | | | |
| 1 : | | | | | | 1 | | | : |
| 29 | | | | | | I | Ī | Ī | 42 |
| 30 | | | | | | Ī | Ī | Ī | Ī |
| max type | 5 | 6 | 10 | 23 | 25 | 31 | 52 | 55 | 65 |

 \perp = No cycle found.

Counter-Examples

Mohar and Thomassen conjecture is false.

The Results

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| Type K_n | K_{15} | K_{16} | K_{19} | K_{27} | K_{28} | K_{31} | K_{39} | K_{40} | K_{43} |
|------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 1 | 8 | 10 | 11 | 12 | 12 | 8 | 12 | 10 | 8 |
| 2 | 11 | 12 | 14 | 16 | 17 | 13 | 15 | 15 | 11 |
| 3 | 12 | 14 | 16 | 19 | 18 | 15 | 20 | 18 | 12 |
| 4 | 13 | 16 | 18 | 20 | | 17 | 24 | 19 | 15 |
| 5 | 14 | 16 | | 27 | 1 | 20 | 26 | 24 | 18 |
| 6 | | 16 | | 1 | 1 | 21 | 30 | 26 | 20 |
| 7 | | | Τ. | | | 23 | 32 | 28 | 21 |
| 8 | | | | 1 | | 24 | | 30 | 23 |
| 9 | | | 1 | 1 | 1 | 28 | | 33 | 24 |
| 10 | | | Τ. | | | 28 | | 35 | 25 |
| 11 | | | | | | 29 | | 36 | 27 |
| 12 | | | | | | | | 38 | 29 |
| 13 | | | | | | Τ. | | 40 | 30 |
| 14 | | | | | | | | | 31 |
| | | | | | | | | | |
| : | | | | | | 1 | \perp | ⊥ | : |
| 29 | | | | | | 1 | | 1 | 42 |
| 30 | | | | | | | | 1 | |
| max type | 5 | 6 | 10 | 23 | 25 | 31 | 52 | 55 | 65 |

 \perp = No cycle found.

Conjecture

For every $\alpha > 0$, there exists a triangulation with no splitting cycles of type larger than $\alpha \cdot \frac{g}{2}$.

Planar Case

Number of

Encoding Toroidal Triangulations

Properties of the planar case:

- 1/ We have a notion of 3-orientation for triangulations.
- 2/ Every 3-orientation admits a unique Schnyder wood coloration.
- 3/ Each color corresponds to a spanning tree and so there is no monochromatic cycle.
- 4/ The 3-orientations of a given triangulation have a structure of distributive lattice.
- 5/ The minimal element of the lattice has no clockwise oriented cycle.
- 6/ Triangulations are in bijection with a particular type of decorated embedded trees.

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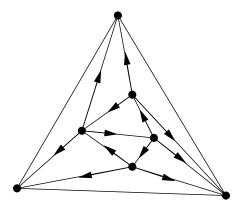
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Conclusion

1/ We have a notion of 3-orientation for triangulations.



Kampen (1976)

Every planar triangulation admits a 3-orientation.

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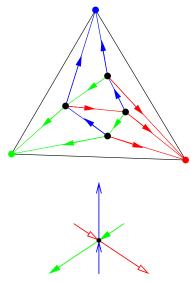
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2/ Every 3-orientation admits a unique Schnyder wood coloration.



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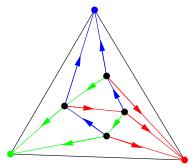
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Conclusion

2/ Every 3-orientation admits a unique Schnyder wood coloration.



de Fraisseix and Ossona de Mendez (2001)

Each 3-orientation of a plane simple triangulation admits a unique coloring (up to permutation of the colors) leading to a Schnyder wood.

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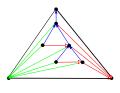
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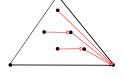
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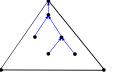
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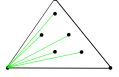
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3/ Each color corresponds to a spanning tree and so there is no monochromatic cycle.









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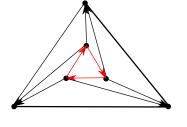
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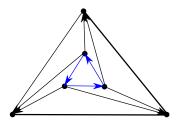
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4/ The 3-orientations of a given triangulation have a structure of distributive lattice.





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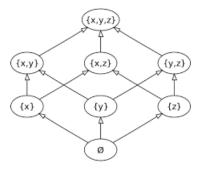
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4/ The 3-orientations of a given triangulation have a structure of distributive lattice.



Propp (1993), Ossona de Mendez (1994), Felsner (2004)

The set of the 3-orientations of a given triangulation has a structure of distributive lattice for the appropriate ordering.

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- 3/ Each color corresponds to a spanning tree and so there is no monochromatic cycle.
- 4/ The 3-orientations of a given triangulation have a structure of distributive lattice.
- 5/ The minimal element of the lattice has no clockwise oriented cycle.
- 6/ Triangulations are in bijection with a particular type of decorated embedded trees.

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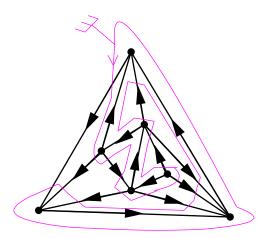
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6/ Triangulations are in bijection with a particular type of decorated embedded trees (Poulalhon and Schaeffer, 2006).



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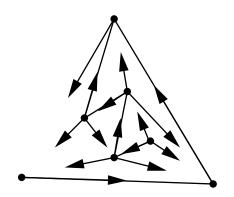
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6/ Triangulations are in bijection with a particular type of decorated embedded trees (**Poulalhon and Schaeffer**, **2006**).



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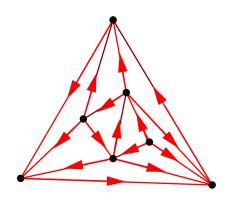
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6/ Triangulations are in bijection with a particular type of decorated embedded trees (**Poulalhon and Schaeffer**, **2006**).



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- 1/ We have a notion of 3-orientation for triangulations.
- 2/ Every 3-orientation admits a unique Schnyder wood coloration.
- 3/ Each color corresponds to a spanning tree and so There is no monochromatic **contractible** cycle.
- 4/ The 3-orientations of a given triangulation have a structure of distributive lattices.
- 5/ The minimal element of each lattice has no clockwise oriented contractible cycle.
- 6/ Triangulations are in bijection with a particular type of decorated **unicellular toroidal maps**.

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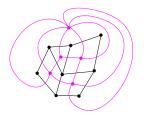
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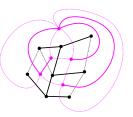
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6/ Triangulations are in bijection with a particular type of decorated **unicellular toroidal maps**.





Tree-cotree Decomposition: (T, C, X). T has n-1 edges, C has f-1 edges and X the remaining. $\chi = n - (n-1+f-1+x) + f \Leftrightarrow x = 2 - \chi = 2q$

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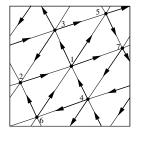
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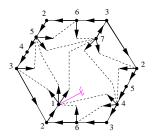
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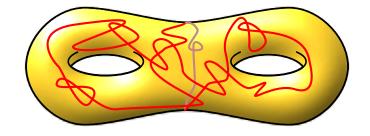
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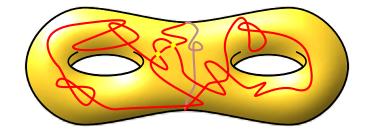
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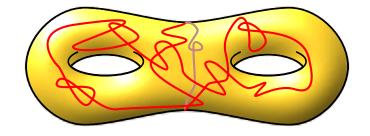
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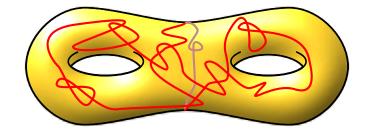
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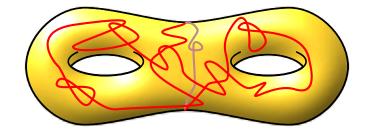
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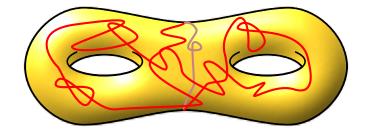
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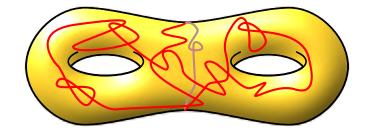
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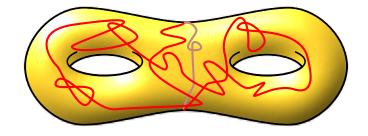
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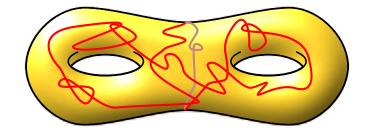
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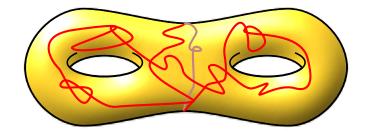
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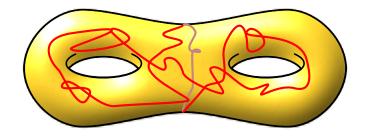


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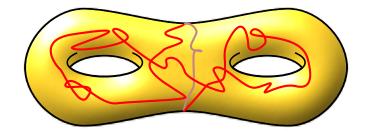
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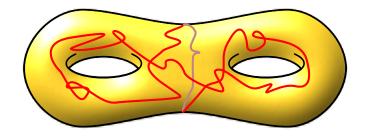
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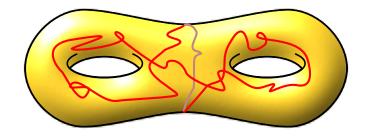
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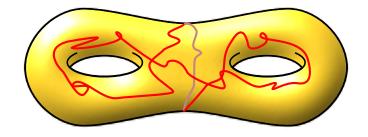
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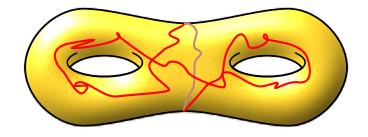
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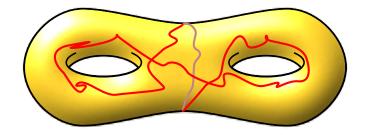
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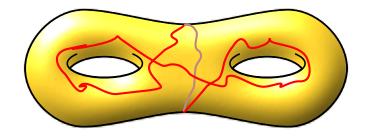
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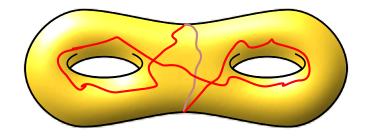
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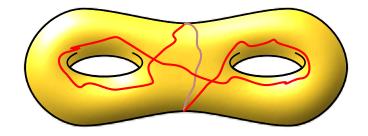
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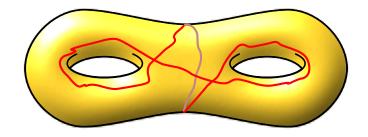
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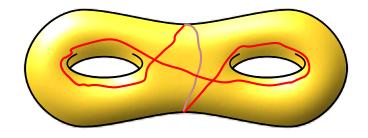
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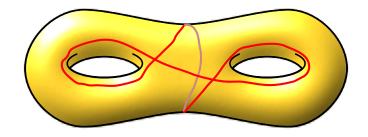
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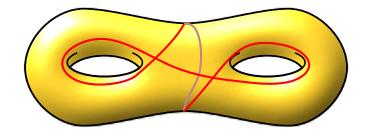
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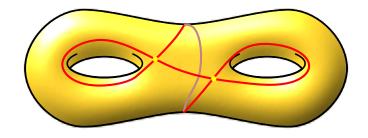
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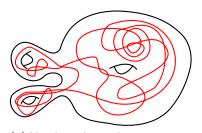
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(g) Number of crossings: too many!

(h) Number of crossings:1 → optimal

Three problems:

- → Deciding if a curve can be made simple by homotopy.
- Finding the minimum possible number of self-intersections.
- → Finding a corresponding minimal representative.

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Boundaries Simple Number Representative $O((g\ell)^2)$ $O((g\ell)^2)$ $O((g\ell)^4)$ b > 0BS (1984) CL (1987) A (2015) b = 0L (1987) L (1987) dGS (1997) $O(\ell^5)$ $O(\ell^5)$ GKZ (2005) GKZ (2005) Any $O(\ell \cdot \log^2(\ell))$

BS: Birman and Series, An algorithm for simple curves on surfaces.

CL: Cohen and Lustig, Paths of geodesics and geometric intersection numbers: I.

L: Lustig, Paths of geodesics and geometric intersection numbers: II.

A: Arettines, A combinatorial algorithm for visualizing representatives with minimal self-intersection.

dGS: de Graaf and Schrijver, Making curves minimally crossing by Reidemeister moves.

GKZ: Gonçalves, Kudryavtseva and Zieschang, An algorithm for minimal number of (self-)intersection points of curves on surfaces.

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Publications:

- 1/ Some Triangulated Surfaces without Balanced Splitting: Published in *Graphs and Combinatorics*.
- 2/ Encoding Toroidal Triangulations: Accepted in *Discrete*& Computationnal Geometry.
- 3/ Computing the Geometric Intersection Number of Curves: Will be submitted to the next SoCG.

Work in progress:

- 1/ Looking for a proof that does not require a computer.
- 2/ There are a lot of implications for the bijection in the plane. Is it possible to generalized them.
- 3/ It remains to look at the construction of a minimal representative for a couple of curves.

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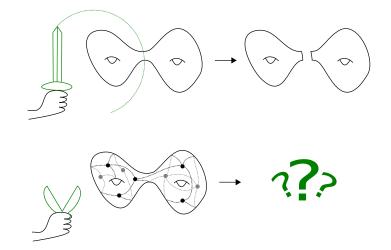
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Conjecture

Deciding if there is a simple closed walk in a given homotopy class is NP-complete and FPT parametrized by the genus of the surface.

Do you have questions?